

A feasible “Kochen-Specker” experiment with single particles.

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We present a simple experimental scheme which can be used to demonstrate an all-or-nothing type contradiction between non-contextual hidden variables and quantum mechanics. The scheme, which is inspired by recent ideas by Cabello and García-Alcaine, shows that even for a single particle, path and spin information cannot be predetermined in a non-contextual way.

Most predictions of quantum mechanics are of a statistical nature, predictions for individual events are probabilistic. The question as to whether one can go beyond quantum mechanics in this respect, i.e. whether there could be hidden variables determining the results of all individual measurements, has been answered to the negative for *local* hidden variables by Bell’s theorem [1]. Locality means that in such theories the results of measurements in a certain space-time region are independent of what happens in a space-time region that is spacelike separated, in particular independent of the settings of a distant measuring apparatus.

Bell’s theorem refers to a situation where there are two particles and where the predictions of quantum mechanics are statistical. Furthermore, even definite (non-statistical) predictions of quantum mechanics are in conflict with a local realistic picture for systems of three particles or more [2].

The Kochen-Specker (KS) theorem [3] states that *non-contextual* theories (NCT) are incompatible with quantum mechanics. Non-contextuality means that the value for an observable predicted by such a theory does not depend on the experimental context, i.e. which other co-measurable observables are measured simultaneously. In quantum mechanics, observables have to commute in order to be co-measurable. Non-contextuality is a more stringent demand than locality because it requires mutual independence of the results for commuting observables even if there is no spacelike separation.

So far there has not been an experimental test of non-contextuality based on the original formulation of the KS theorem, which refers to a single spin-1 particle (cf. [4]). However, experimental tests of local hidden variable theories, such as tests of Bell’s inequality and of the GHZ

paradox [2], can also be seen as tests of NCT. Note that such experiments in general involve several particles.

Recently, in a very interesting paper Cabello and García-Alcaine (CG) [5] have proposed an experimental test of the KS theorem based on two two-level systems (qubits).

In this paper we present a simple experimental scheme to test non-contextuality which is inspired by the CG argument. The experiment can be realized with single particles, using both their path and their spin degrees of freedom. It leads to a non-statistical test of non-contextuality versus quantum mechanics. In this respect it is similar to the GHZ argument against local realism.

In the following, we first show how a very direct experimental test of non-contextuality can be found, then we discuss our operational realization.

Consider four binary observables Z_1, X_1, Z_2 , and X_2 . Let us denote the two possible results for each observable by ± 1 . In a NCT these observables have predetermined non-contextual values $+1$ or -1 for individual systems, denoted as $v(Z_1), v(Z_2), v(X_1)$, and $v(X_2)$. This means e.g. that for an individual system the result of a measurement of Z_1 will always be $v(Z_1)$ irrespective of which other co-measurable observables are measured with it. We will show that the existence of such non-contextual values is incompatible with quantum mechanics.

Imagine an ensemble E of systems for which one always finds equal results for Z_1 and Z_2 , and also for X_1 and X_2 . (Clearly, in order for this statement to be meaningful, Z_1 and Z_2 , and X_1 and X_2 have to be co-measurable.) In a NCT this means that

$$v(Z_1) = v(Z_2) \quad \text{and} \quad v(X_1) = v(X_2) \quad (1)$$

for each individual system of the ensemble. Then there are only two possibilities: either $v(Z_1) = v(X_2)$, which implies $v(X_1) = v(Z_2)$; or $v(Z_1) \neq v(X_2)$, which implies $v(X_1) \neq v(Z_2)$. We will see that this elementary logical deduction is already sufficient to establish a contradiction between NCT theories and quantum mechanics.

To this end, let us express the above argument in a slightly different way. Eq. (1) can be written as

$$v(Z_1)v(Z_2) = v(X_1)v(X_2) = 1. \quad (2)$$

Multiplying by $v(X_2)v(Z_2)$ it immediately follows that

$$v(Z_1)v(X_2) = v(X_1)v(Z_2). \quad (3)$$

Let us now introduce the notion of product observables such as $Z_1 X_2$. By definition, one way of measuring $Z_1 X_2$ is to measure Z_1 and X_2 separately and multiply the results; in general, there are other ways. In particular, if another compatible observable (e.g. $X_1 Z_2$, cf. below) is measured simultaneously, it will in general not be possible to obtain separate values for Z_1 and X_2 . However, in a non-contextual theory, the result of a measurement of an observable must not depend on which other observables are measured simultaneously. Therefore the predetermined value $v(Z_1 X_2)$, for example, in a NCT has to follow the rule [5]

$$v(Z_1 X_2) = v(Z_1) v(X_2). \quad (4)$$

In this new language, our above argumentation can be resumed in the following way:

$$v(Z_1 Z_2) = v(X_1 X_2) = 1 \Rightarrow v(Z_1 X_2) = v(X_1 Z_2) \quad (5)$$

i.e. if our systems have the property expressed in Eq. (1), then the two product observables $Z_1 X_2$ and $X_1 Z_2$ must always be equal in a NCT. Note that in general this prediction of NCT can only be tested if $Z_1 X_2$ and $X_1 Z_2$ are co-measurable.

It follows from the results of [5] that the prediction (5) leads to an observable contradiction with quantum mechanics. To see this, consider a system of two qubits and the observables [5]

$$Z_1 := \sigma_z^{(1)}, X_1 := \sigma_x^{(1)}, Z_2 := \sigma_z^{(2)}, X_2 := \sigma_x^{(2)}, \quad (6)$$

where $\sigma_z^{(1)}$ means the z-component of the “spin” of the first qubit etc. It is easy to check that this set of observables satisfies all the properties required above. In particular, while Z_1 and X_1 , and Z_2 and X_2 , do not commute, the two product observables $Z_1 X_2$ and $X_1 Z_2$ do. Furthermore, the quantum-mechanical two-qubit state

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+x\rangle|+x\rangle + |-x\rangle|-x\rangle) \end{aligned} \quad (7)$$

is a joint eigenstate of the commuting product observables $Z_1 Z_2$ and $X_1 X_2$ with both eigenvalues equal to +1. Therefore, on the one hand the ensemble described by this state possesses the property of the ensemble E discussed above (cf. (1)): the measured values of $Z_1 Z_2$ and $X_1 X_2$ are equal to +1 for every individual system. On the other hand, quantum mechanics predicts for the state $|\psi_1\rangle$, that the measured value of $Z_1 X_2$ will always be opposite to the value of $X_1 Z_2$. This can be seen by decomposing $|\psi_1\rangle$ in the basis of the joint eigenstates of the two commuting product observables $Z_1 X_2$ and $X_1 Z_2$:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\chi_{1,-1}\rangle + |\chi_{-1,1}\rangle), \quad (8)$$

with

$$\begin{aligned} |\chi_{1,-1}\rangle &= \frac{1}{2}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle \\ &\quad + |+z\rangle|-z\rangle + |-z\rangle|+z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+z\rangle|+x\rangle - |-z\rangle|-x\rangle) \\ &= \frac{1}{\sqrt{2}}(|-x\rangle|+z\rangle + |+x\rangle|-z\rangle) \end{aligned} \quad (9)$$

$$\begin{aligned} |\chi_{-1,1}\rangle &= \frac{1}{2}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle \\ &\quad - |+z\rangle|-z\rangle + |-z\rangle|+z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+z\rangle|-x\rangle + |-z\rangle|+x\rangle) \\ &= \frac{1}{\sqrt{2}}(|+x\rangle|+z\rangle - |-x\rangle|-z\rangle). \end{aligned} \quad (10)$$

and

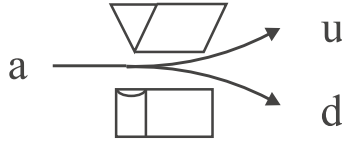
$$\begin{aligned} Z_1 X_2 |\chi_{1,-1}\rangle &= +|\chi_{1,-1}\rangle \\ X_1 Z_2 |\chi_{1,-1}\rangle &= -|\chi_{1,-1}\rangle \\ Z_1 X_2 |\chi_{-1,1}\rangle &= -|\chi_{-1,1}\rangle \\ X_1 Z_2 |\chi_{-1,1}\rangle &= +|\chi_{-1,1}\rangle \end{aligned} \quad (11)$$

From (8) and (11) one sees that $|\psi_1\rangle$ is a linear combination of exactly those joint eigenstates of $Z_1 X_2$ and $X_1 Z_2$ for which the respective eigenvalues are opposite, which means, of course, that in a joint measurement the two observables will always be found to be different. With Eq. (5) in mind, this implies that the ensemble described by $|\psi_1\rangle$ cannot be described by any non-contextual theory.

Note that one would already have a contradiction if quantum mechanics only predicted that the observed values of $Z_1 X_2$ and $X_1 Z_2$ are sometimes different, but in fact the result is even stronger, with QM and NCT predicting exactly opposite results. Thus, we have conflicting predictions for observable effects on a non-statistical level [6] (cf. [2]).

According to the argument presented in the previous paragraph, an experimental test of non-contextuality can be performed in the following way: (i) Show that $Z_1 Z_2 = 1$ and $X_1 X_2 = 1$ for systems prepared in a certain way. (ii) Determine whether $Z_1 X_2$ and $X_1 Z_2$ are equal for such systems. Note that in steps (i) and (ii) the observables Z_1, X_1, Z_2 , and X_2 appear in two different contexts.

Quantum mechanics predicts that step (i) can be accomplished by constructing a source of systems described by the state $|\psi_1\rangle$ and measuring $Z_1 Z_2$ and $X_1 X_2$ on these systems. According to QM, both $Z_1 Z_2$ and $X_1 X_2$ will always be found to be equal to +1. This can e.g. be verified by measuring the pairs Z_1 and Z_2 and X_1 and X_2 separately on many systems, and obtaining the values of $Z_1 Z_2$ and $X_1 X_2$ by multiplication. Alternatively, one could also perform joint measurements of $Z_1 Z_2$ and



$$|a\rangle|x+\rangle \rightarrow \frac{1}{\sqrt{2}}(|u\rangle|z+\rangle + |d\rangle|z-\rangle)$$

FIG. 1. Possible way of creating the single-particle version of $|\psi_1\rangle$ given in Eq. (12) using a standard Stern-Gerlach apparatus. A single particle with spin state $|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$, i.e. spin along the positive x direction, comes in from the left (spatial mode $|a\rangle$). By the Stern-Gerlach device, which separates incoming states according to the z -components of their spin, this is transformed into the desired superposition state. The outputs u and d could be connected to the inputs of the devices of Figures 2 or 3

X_1X_2 on individual systems, but for step (i) such joint measurements are not strictly necessary. On the other hand, step (ii) definitely requires a joint measurement of Z_1X_2 and X_1Z_2 , because both negative and positive values are to be expected for Z_1X_2 and X_1Z_2 , and we have to determine whether their values are equal (as required by NCT) or opposite (as predicted by QM) for individual systems.

One might argue that the above alone is sufficient to demonstrate in a very direct way the contextuality of quantum mechanics and that, in view of the numerous experiments confirming the predictions of quantum mechanics concerning entangled states [7] of the kind of Eq. (7), an experiment may not be necessary. Nevertheless, an explicit operational realization could have instructive advantages. Therefore we now discuss a simple and intuitive experiment which would be readily realizable.

One could consider realizing the above protocol with two entangled particles, each one representing one of the qubits. Yet when considering contextuality, non-locality is not an issue. This is underlined by the observation that our contradiction arises for joint measurements of the two qubits. Therefore the two necessary qubits can also be carried by the same single particle.

In our scheme, the first qubit is emulated by the spatial modes of propagation (paths) of a single spin-1/2 particle or photon, and the second qubit by its spin (or polarization) degree of freedom [8]. Spin-1/2 and photon polarization are completely equivalent for our purposes. Our setup requires a source of polarized single particles, beam splitters, and Stern-Gerlach type devices. In practice, the experiment would be easiest to do with photons because all these elements are readily available, in particular polarized single-photon states can be produced to excellent approximation via parametric down-conversion [9]. Nevertheless, we will use the spin language in the sequel because it is more familiar to most physicists.

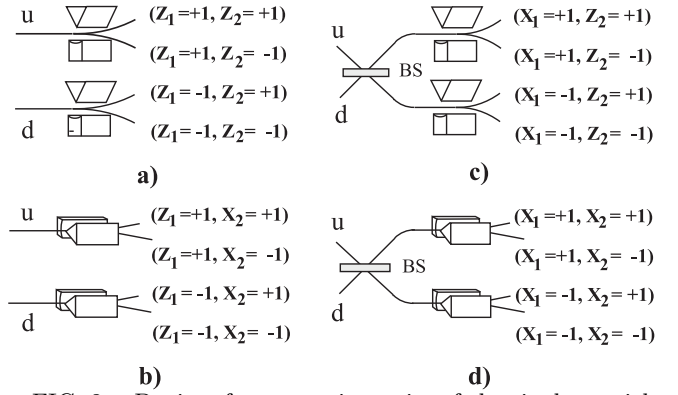


FIG. 2. Devices for measuring pairs of the single-particle observables of Eq. (13). A particle comes in from the left. Note that in general the incoming states will have components in both spatial modes u and d and of different spin. The devices shown measure: a) Z_1 and Z_2 ; b) Z_1 and X_2 ; c) X_1 and Z_2 ; d) X_1 and X_2 . BS in c) and d) stands for a 50–50 beam-splitter (see main text), which changes the basis of path analysis from $|u\rangle, |d\rangle$, corresponding to a measurement of Z_1 , to $|u'\rangle, |d'\rangle$, thus leading to a measurement of X_1 . In a) and c) the Stern-Gerlach apparatus are oriented along the z -axis (measurement of Z_2), in b) and d) along the x -axis (measurement of X_2).

Consider a situation where the particle can propagate in two spatial modes u and d , and let $|z+\rangle, |z-\rangle$ etc. denote the particle's spin states as before. Then the state $|\psi_1\rangle$ of Eq. (7) is mapped onto the one-particle state

$$\frac{1}{\sqrt{2}}(|u\rangle|z+\rangle + |d\rangle|z-\rangle). \quad (12)$$

In Fig. 1 we show how such a state can be prepared experimentally.

The observables Z_1, X_1, Z_2, X_2 are now represented by

$$\begin{aligned} Z_1 &= |u\rangle\langle u| - |d\rangle\langle d| \\ X_1 &= |u'\rangle\langle u'| - |d'\rangle\langle d'| \\ Z_2 &= |z+\rangle\langle z+| - |z-\rangle\langle z-| \\ X_2 &= |x+\rangle\langle x+| - |x-\rangle\langle x-|, \end{aligned} \quad (13)$$

where $|u'\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$, $|d'\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$, $|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$, $|x-\rangle = \frac{1}{\sqrt{2}}(|z+\rangle - |z-\rangle)$, i.e. u' and d' denote the output modes of a 50-50 beam-splitter with inputs u and d , and $|x+\rangle$ and $|x-\rangle$ are the spin eigenstates along the x direction. Clearly, Z_1 and X_1 act on the path, and Z_2 and X_2 on the spin degree of freedom.

In Fig. 2 we show the devices that measure pairs of one-particle observables, such as Z_1 and Z_2 . While any device that performs a state analysis in the basis of common eigenstates of Z_1X_2 and X_1Z_2 can be considered to perform a joint measurement of these two observables, the particular realization presented in Fig. 3 has the merit of showing explicitly that a joint measurement of two product observables is performed. It also

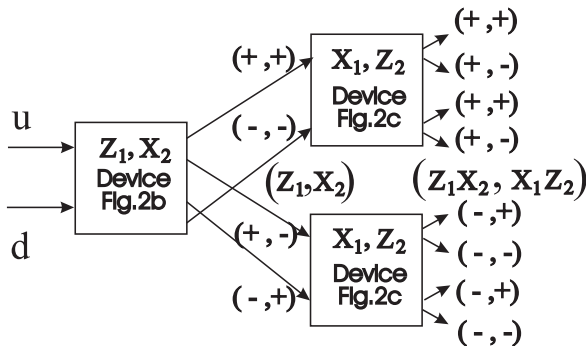


FIG. 3. Device for performing a joint measurement of Z_1X_2 and X_1Z_2 . A device performing a joint measurement of Z_1X_2 and X_1X_2 can be constructed in an analogous way. Instead of leading to detectors, the outputs of the device of Fig. 2b, which measures Z_1 and X_2 (the connecting beams are labeled by the signs of the eigenvalues of the two observables), are now fed into two replicas of the device of Fig. 2c, which measure X_1 and Z_2 . The output beams are labeled by the signs of the eigenvalues of Z_1X_2 and X_1Z_2 . The first device separates the two eigenspaces of the degenerate product observable Z_1X_2 . Eigenstates of Z_1X_2 with eigenvalue +1 are sent up, those with eigenvalue -1 are sent down. Detection of the particle behind one of the two subsequent devices is a measurement of X_1Z_2 and at the same time completes the measurement of Z_1X_2 . For an ensemble of particles with the property $Z_1X_2 = X_1X_2 = 1$ (which can be verified using the devices of Fig. 2a and 2d) quantum mechanics predicts that the particles can emerge only via one of those four outputs for which the values of Z_1X_2 and X_1Z_2 are *opposite*, i.e. the second, fourth, fifth, and seventh from the top, whereas non-contextual theories predict exactly the complementary set of outputs.

shows how the information that is obtained in the first stage of the measurement about the values of Z_1 and X_2 is partially erased such that only information about the product Z_1X_2 is retained, in order to make the measurement of X_1Z_2 in the second stage possible. Using the devices of Figs. 1-3 one can realize the full experimental procedure described above.

The present scheme allows the simplest non-statistical experimental test of non-contextuality that is known to us [10]. Similarly to the original Kochen-Specker paradox it requires only a single particle (though two degrees of freedom). With the experimental setup consisting of a simple interferometer, it shows particularly clearly that the appearance of the paradox is related to the superposition principle.

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- [1] J.S. Bell, *Physics* **1**, 195 (1965).
 - [2] D.M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989); D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, *Am. J. Phys.* **58**, 1131 (1990)
 - [3] E.P. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990); S. Kochen and E.P. Specker, *J. Math. Mech.* **17**, 59 (1967); J.S. Bell, *Rev. Mod. Phys.* **38**, 447 (1966); A. Peres, *J. Phys. A* **24**, L175 (1991); N.D. Mermin, *Rev. Mod. Phys.* **65**, 803 (1993). See also A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993).
 - [4] C. Simon, Č. Brukner, and A. Zeilinger, quant-ph/0006043; see also N.D. Mermin, quant-ph/9912081
 - [5] A. Cabello and G. García-Alcaine, *Phys. Rev. Lett.* **80**, 1797 (1998). The CG version of the KS theorem also has the distinguishing feature of being state independent, in the spirit of the original Kochen-Specker theorem.
 - [6] Of course, in a real experiment visibilities are never perfect, and one would have to use some kind of inequality to rigorously establish the contradiction.
 - [7] A. Zeilinger, *Rev. Mod. Phys.* **71**, S288 (1999)
 - [8] M. Zukowski, *Phys. Lett. A* **157**, 198 (1991); M. Czachor, *Phys. Rev. A* **49**, 2231 (1994).
 - [9] S. Friberg, C.K. Hong, and L. Mandel, *Phys. Rev. Lett.* **54**, 2011 (1985)
 - [10] For a single-photon experiment that implements a statistical test of NCT versus QM see M. Michler, H. Weinfurter, and M. Zukowski, *Phys. Rev. Lett.* **84**, 5457 (2000).